Note \( c = 3 \times 10^8 \text{m/s}, \ h = 1.055 \times 10^{-34} \text{Js}, \ m_e = 9.11 \times 10^{-31} \text{kg}, \ q = 1.6 \times 10^{-19} \text{C}, \ 
\varepsilon_0 = 8.85 \times 10^{-12} \text{F/m}, \ k = 1.38 \times 10^{-23} \text{J/K}. \)

30 pts.

1. Calculate the commutator \([E,x]\) and explain what the outcome implies for the simultaneous measurement of \( E \) and \( x \).

40 pts.

2a. Explain why we use the concept of wavepackets to describe the behavior of particles.
2b. Explain the concept of group velocity.
2c. The \( E(k) \) relationship of the conduction band close to \( E_c \) can be described as \( E(k) = \frac{\hbar^2 k^2}{2m} \). Derive an expression for the group velocity for this case.
2d. The \( E(k) \) relationship for higher electron energies approaches a linear form given by \( \alpha E^2 = \frac{\hbar^2 k^2}{2m} \), where \( \alpha \) is a constant equal to about \( 1/E_g \), where \( E_g \) is the bandgap of the semiconductor considered. Again, derive an expression for the electron group velocity in this case.

30 pts.

3a. Calculate the two lowest energy levels \( E_1 \) and \( E_2 \) of a one dimensional, square quantum well device with infinite barriers if \( L = 100 \text{Angstrom} \) within the free electron model. Express your answer in eV.
3b. What is the wavelength of light emission (in \( \mu \text{m} \)) when an electron drops from \( E_2 \) into \( E_1 \).
1. $[E, x] = ?$

$[E, x] = E \times \dot{\Phi} - x \times E \cdot \Phi$

with $E, x$ operators.

$\dot{\Phi} = \psi(x), \psi(t)$

$E = \frac{\partial \Phi}{\partial t}, \quad x = x$

$\Rightarrow$ first term: $E \times \dot{\Phi} = E \times \psi(x) \times \dot{\Phi}$

$= x \psi(x), \frac{\partial \Phi}{\partial t} = \frac{\partial x \times \psi(x)}{\partial t}$

Second term: $x \times E \cdot \dot{\Phi} = x \times \frac{\partial \Phi}{\partial t} (\psi(x) \cdot \dot{\Phi})$

$= x \times \frac{\partial x \times \psi(x)}{\partial t} = \frac{\partial x \times \psi(x)}{\partial t}$

Hence $[E, x] = \frac{\partial x \times \psi(x)}{\partial t} \frac{\partial (\psi(x) \cdot \dot{\Phi})}{\partial t} = \frac{\partial (\psi(x) \cdot \dot{\Phi})}{\partial t} = 0$

$\Rightarrow E$ and $x$ commute: not subject to Heisenberg uncertainty principle, which implies that $E$ and $x$ can be measured simultaneously with an accuracy only limited by equipment, not by the physical constraints of the system.
Wave packets are necessary to describe the phenomenon of localization. Although wave modules squared, extend uniformly over all of space.

b) Group velocity is defined as the velocity of the wave packet, rather than of the individual constituent waves. \( V_g \neq \frac{d\omega}{dk} \) but \( \frac{dV}{dk} \).

\[
E = \frac{\hbar^2 k^2}{2m}, \quad \text{note: } E = \hbar \omega \Rightarrow \frac{dE}{dk} = \hbar \frac{d\omega}{dk} = \hbar \frac{dV}{dk}.
\]

\[
V_g = \frac{1}{h} \frac{dE}{dk} = \frac{1}{h} \left( \frac{\hbar^2 k^2}{2m} \right) = \frac{\hbar k}{2m}, \quad \text{where} \quad V_g = \frac{\hbar k}{2m}.
\]

Which makes sense since we know \( \frac{d\hbar k}{dk} = \frac{V_g}{m} = \frac{\hbar k}{m} \) classically.

\[
aE^2 = \frac{\hbar^2 k^2}{2m} \quad \Rightarrow \quad E^2 = \frac{\hbar^2 k^2}{2m} \quad \Rightarrow \quad E = \left( \frac{\hbar k}{2m} \right)^{\frac{1}{2}}.
\]

\[
V_g = \frac{1}{h} \frac{dE}{dk} = \frac{1}{h} \frac{1}{2} \left( \frac{\hbar^2 k^2}{2m} \right)^{-\frac{1}{2}} \cdot \frac{\hbar^2}{2m} \cdot 2k = \frac{\hbar k}{2m}.
\]

\[
V_g = \left( \frac{2m \lambda}{\hbar} \right)^{\frac{1}{2}}, \quad \text{which with } m = m_0 = 9.11 \times 10^{-31} \text{kg}
\]

\[
\lambda = \frac{\hbar}{E} \Rightarrow E = \frac{m_0 \hbar^2}{\lambda^2} = 10^{-1} \text{eV}.
\]

\[
\Rightarrow V_g = 3 \times 10^7 \text{cm/s} \text{ close to the measured velocity at saturation velocity.}
\]
Stationary boundary condition

\[ k_x = \frac{N_x \pi}{L}, \quad N_x = 1, 2 \text{ etc.} \]

\[ E = \frac{\hbar^2 k_x^2}{2m} \]

\[ \Rightarrow E = \frac{\hbar^2}{2m} \left( \frac{N_x \pi}{L} \right)^2 \]

\[ E_1 = \frac{\hbar^2}{2m} \left( \frac{1}{L} \right)^2 = \left( \frac{1.055 \times 10^{-34}}{2 \times 9.11 \times 10^{-31}} \right)^2 \left( \frac{1}{100 \times 10^{-10}} \right)^2 \]

\[ E_1 = 6.03 \times 10^{-22} \text{ eV} = \frac{6.03 \times 10^{-22}}{1.6 \times 10^{-19}} = 3.77 \text{ meV} \]

\[ E_2 = 4 E_1 = 15.1 \text{ meV}. \]

\[ E_2 - E_1 = h \nu = \frac{hc}{\lambda} \]

\[ \lambda = \frac{hc}{E_2 - E_1} = \frac{1.055 \times 10^{-34} \times 2 \times 12 \times 3 \times 10^9}{(15.1 - 3.77) \times 1.6 \times 10^{-19}} \]

\[ \lambda = 1.0 \times 10^{-4} \text{ m} = 110 \text{ nm} \]